# Approximating a point using interpolating polynomials

Note that in this problem set, you may be off in the last digit or two, which is okay, but if your error is more significant than that, please speak to your peers, a teaching assistant or the instructor.

1. Given the two points (2517.32, 13.92) and (2527.54, 18.23), approximate the value at $x = 2522.91$ by first finding $h$ and $\delta$, then finding the appropriate interpolating polynomial, and then evaluate this polynomial at this $\delta$.

Answer: $h = 10.22$, and we shift and scale 2517.32 to –0.5 and 2527.54 to 0.5. Thus, the midpoint is $m = 2522.43$ so $\delta = (x - m)/h = 0.04696673189823875$, and the polynomial is $4.31\delta + 16.075$, and evaluating this at $\delta = 0.04696673189823875$ yields 16.27742661448141.

2. Given the three points (2517.32, 13.92), (2527.54, 18.23) and (2537.76, 19.61), approximate the value at $x = 2524.81$ by first finding $h$ and $\delta$, then finding the appropriate interpolating polynomial, and then evaluate this polynomial at $\delta$.

Answer: $h = 10.22$, and because $x$ is between the first two points, we shift and scale the 2517.32 to –0.5, 2527.54 to 0.5, and thus the third point, 2537.76, is mapped to 1.5. The midpoint is $m = 2522.43$.

Thus, $\delta = (x - m)/h = 0.2328767123287671$, the polynomial is $-1.465x^2 + 4.31x + 16.44125$, and evaluating this at $\delta = 0.2328767123287671$ yields 17.36549939012948.

3. Given the four points (2517.32, 13.92), (2527.54, 18.23), (2537.76, 19.61) and (2547.98, 20.04), approximate the value at $x = 2537.09$ by first finding $h$ and $\delta$, then finding the appropriate interpolating polynomial, and then evaluate this polynomial at $\delta$.

Answer: $h = 10.22$, and because $x$ is between the two middle values, we shift and scale 2527.54 to –0.5 and 2537.76 to 0.5, and thus we shift and scale 2517.32 to –1.5 and 2547.98 to 1.5. The midpoint is $m = 2532.65$, and thus $\delta = (x - m)/h = 0.4344422700587084$, the polynomial is $0.33x^3 - 0.97x^2 + 1.2975x + 19.1625$, and evaluating this at $\delta = 0.4344422700587084$ yields 19.57016986353606.

4. What is the purpose of shifting and scaling the $x$ values?

Answer: To reduce numerical error by reducing the condition number of the Vandermonde matrix and reduce the impact of subtractive cancellation when evaluating the resulting polynomial. If you were to simply find the interpolating polynomial in Question 3 without shifting and scaling, evaluating that interpolating polynomial at the point $x$ would yield 19.570169863. Contrast the correct answer to 20 digits, the approximation found in Question 3, and this answer:

19.570169863536058828
19.57016986353606
19.57016986300000

Your computer may get slightly different results, but the magnitude of the errors will be approximately the same.

Given the three points (2517.32, 13.92), (2527.54, 18.23) and (2537.76, 19.61),

5. Repeat Question 2, but evaluate the polynomial at $x = 2534.81$ under the assumption that the last point is our most recent reading (so no data is to the right of the last point).

Answer: $h = 10.22$, but now the point is between the last two points, so we shift and scale 2527.54 to – 0.5 and 2537.76 to 0.5, and 2517.32 is shifted and scaled to –1.5. The midpoint is $m = 2532.65$ and thus $\delta = (x - m)/h = 0.2113502935421$, the polynomial is $-1.465x^2 + 1.38x + 19.28625$, and evaluating this at $\delta = 0.2113502935421$ yields 19.51247339834791.

6. Repeat Question 3, but evaluate the polynomial at $x = 2538.12$ under the assumption that the last point is our most recent reading (so no data is to the right of the last point).

Answer: $h = 10.22$, but now the point is between the last two points, so we shift and scale the last two points to –0.5 and 0.5, respectively, and the first two points are shifted and scaled to –2.5 and –1.5, respectively. Thus, $\delta = -0.4647749510763209$, the polynomial is $0.33x^2 + 0.02x^2 + 0.3475x + 19.82$, and evaluating this at $\delta = -0.4647749510763209$ yields 19.62967944463383.

7. Why do we require that $-0.5 < \delta < 0.5$?

Answer: All of these calculations are based on the assumption that we are sampling the $x$ values uniformly, and therefore if $\delta$ is outside these bounds, then there should be another point around which we should be evaluating the polynomial.

8. We have not discussed the error of an interpolating polynomial. It can be shown that the error for a polynomial interpolating the points $(x_0, f(x_0)) \ldots, (x_n, f(x_n))$ which is being evaluated at a point $x$ has an error that is interpolating a function $f(x)$ which is at least $n + 1$ times differentiable is

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n).$$

Questions: What is the maximum absolute values of each of the following if $-0.5 < \delta < 0.5$?

$(\delta - 0.5)(\delta + 0.5)$
$(\delta - 1)\delta(\delta + 1)$
$(\delta - 1.5)(\delta - 0.5)(\delta + 0.5)(\delta + 1.5)$
$(\delta - 1.5)(\delta - 0.5)(\delta + 0.5)$
$(\delta - 2.5)(\delta - 1.5)(\delta - 0.5)(\delta + 0.5)$

Answer: Using calculus or plotting, we have:

$|(\delta + 0.5)(\delta - 0.5)| \le 0.25$
$|(\delta + 1)\delta(\delta - 1)| \le 0.375$
$|(\delta + 1.5)(\delta + 0.5)(\delta - 0.5)(\delta - 1.5)| \le 0.5625$
$|(\delta + 1.5)(\delta + 0.5)(\delta - 0.5)| \le 0.38490018$
$|(\delta + 2.5)(\delta + 1.5)(\delta + 0.5)(\delta - 0.5)| \le 1$

9. We have shifted and scaled the $x$ values, so these maximum errors should be multiplied by $h$ raised to the corresponding power. Consequently, so long as we ensure that $-0.5 < \delta < 0.5$, what is the best description of the maximum error of each of these formulas?

Answer: The maximum error would thus be $0.25\frac{f^{(2)}(\xi)}{2!}h^2$, $0.375\frac{f^{(3)}(\xi)}{3!}h^3$, $0.5625\frac{f^{(4)}(\xi)}{4!}h^4$, $0.38490018\frac{f^{(3)}(\xi)}{3!}h^3$ and $\frac{f^{(4)}(\xi)}{4!}h^4$ where $\xi$ is some value between the minimum and maximum $x$ values.

10. How do these compare to Taylor series?

Answer: As long as we require that $-0.5 < \delta < 0.5$, these are better than or equal to a corresponding Taylor series of the same power.